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Stress formulation in 3D elasticity and application to spherically uniform anisotropic solids

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Abstract

We formulate the boundary value problem of traction for inhomogeneous anisotropic elastic materials in terms of stresses following the method introduced by Pobedria and apply it to spherically anisotropic materials. An example of spherically symmetric deformation of spherically uniform anisotropic materials is presented.

Keywords: Stress; Elasticity; Anisotropic; Boundary value

1. Introduction

This paper illustrates the stress formulation for linear elastic inhomogeneous anisotropic solids in the particular case of spherical uniform anisotropy. Recently, inhomogeneous materials have attracted interest due to the wide applicability of functionally graded materials (FGM), which are characterized by a microscale that is spatially variable on a macroscale. Inhomogeneous materials exhibit interesting physical properties such as those observed in particular examples of radially dependent isotropic and spherically uniform anisotropic materials by Horgan and Chan (1999a,b), Horgan and Baxter (1996) and Ting (1999). These unexpected properties are that the hoop stress in a spherical shell may not achieve its maximum on the inner boundary and the stresses in an infinite body with a traction-free spherical cavity cannot be obtained as a limit of those of a spherical shell. It was observed that, for a solid sphere, cavitation occurs in the center even for the slightest degree anisotropy.

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The above solutions were obtained by using the displacement formulation of the boundary value problem of traction. Problems of elasticity can be posed in terms of displacement or stresses. The displacement formulation gives well-defined boundary value problems. This is the main reason it is used by applied mathematicians and numerical analysts. However, convergence of numerical algorithms is worse if one is interested in the approximation of the derivatives of the displacement field. Thus, on practice the determination of the stress field leads to a loss of numerical accuracy. On the other hand, a well-defined stress formulation is not obvious. In 3D case the stresses satisfy nine equations in the interior of the body (three equations of equilibrium and six equations following from the compatibility of strains) and only three boundary conditions.

A well-posed formulation in terms of stresses for 3D problems with given traction was proposed by Pobedria (1980, 1979). He observed that it is sufficient to use the equilibrium equations only as boundary conditions. That gives six boundary conditions for the six equations in the body. The equivalence of the displacement and stress formulation was further studied by Kucher et al. (2004), while Li et al. (in press), further refined the theory and obtained new conservation laws based on the stress formulation exploiting the symmetries of the compatibility equations.

In this paper we illustrate the stress method by considering the spherically symmetric problem for spherically uniform anisotropic elastic solids. The solution of this problem in displacement form can be found in the book of Lekhnitskii (1963); some properties of this solution were recently analyzed by Horgan and Baxter (1996) and Ting (1999) by the displacement methods.

2. Displacement and stress formulation of inhomogeneous elastic problems with given boundary traction

Consider the inhomogeneous elastic material with constitutive relation

$$\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{\varepsilon}), \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}$ is the tensor of linear deformation

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.2}$$

Then the displacement formulation of the problem of traction is

$$C_{ij,j}(\mathbf{x}, \boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{x}))) = -F_i \quad \text{in } \Omega, \tag{2.3}$$

$$C_{ij}(\mathbf{x}, \boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{x})))n_j = t_j \quad \text{on } \partial\Omega. \tag{2.4}$$

where $\mathbf{F}$ is a vector of the body forces, $\mathbf{t}$ is the traction vector at the boundary $\partial\Omega$ with outward normal $\mathbf{n}$.

Assume that the constitutive relation (2.1) can be inverted

$$\boldsymbol{\varepsilon} = \mathbf{S}(\mathbf{x}, \boldsymbol{\sigma}), \tag{2.5}$$

Then the compatibility of the linear strain, equilibrium in the domain and the traction boundary condition take, respectively, the following form

$$\Delta S_{ij}(\mathbf{x}, \boldsymbol{\sigma}) + S_{kk,ij}(\mathbf{x}, \boldsymbol{\sigma}) - S_{ik,kj}(\mathbf{x}, \boldsymbol{\sigma}) - S_{jk,ki}(\mathbf{x}, \boldsymbol{\sigma}) = 0 \quad \text{in } \Omega. \tag{2.6}$$

$$\sigma_{ik,k} + F_i = 0 \quad \text{in } \Omega, \tag{2.7}$$

$$\sigma_{ik}n_k = t_i \quad \text{on } \partial\Omega \tag{2.8}$$

Thus, one has 9 equations in Ω and only 3 boundary conditions on the boundary $\partial\Omega$.

We develop below the formulation of a well-defined boundary value problem of traction for the stress tensor—as introduced in Pobedria (1979)—to inhomogeneous elastic solids. The equations in the domain are

$$\Delta S_{ij}(\mathbf{x},\boldsymbol{\sigma}) + S_{kk,ij}(\mathbf{x},\boldsymbol{\sigma}) - S_{ik,kj}(\mathbf{x},\boldsymbol{\sigma}) - S_{jk,ki}(\mathbf{x},\boldsymbol{\sigma}) + Z_{ij}(S_{kl,kl}(\mathbf{x},\boldsymbol{\sigma}) - \Delta S_{kk}(\mathbf{x},\boldsymbol{\sigma})) + R_{ij}(\mathbf{x},\mathbf{Q}(\mathbf{F},\boldsymbol{\sigma})) + (Z_{ij} - \delta_{ij})R_{kk}(\mathbf{x},\mathbf{Q}(\mathbf{F},\boldsymbol{\sigma})) = 0 \quad \text{in } \Omega. \tag{2.9}$$

where $\mathbf{Z}$ is a constant tensor, components of the tensor $\mathbf{Q}$ are the first order derivatives of the left hand side of the equilibrium equation defined by

$$Q_{ij} = \frac{1}{2}(q_{i,j} + q_{j,i}),$$
$$q_i = \sigma_{ik,k} + F_i,$$

and $\mathbf{R}$ is a symmetric tensor function. Six boundary conditions are defined by

$$\sigma_{ik,k} = -F_i \text{ on } \partial\Omega, \tag{2.10}$$

$$\sigma_{ik}n_k = t_i \text{ on } \partial\Omega, \tag{2.11}$$

It is easy to see that if the tensor function $\mathbf{R}$ satisfies the condition $R_{ij}(\mathbf{x},0) \equiv 0$, any solution of Eqs. (2.6)–(2.8) is a solution of Pobedria's system (2.9)–(2.11).

On the other hand, under additional assumptions on $\mathbf{R}$ the converse is also true. Indeed, taking the trace and divergence of Eq. (2.9) we get

$$(2 - Z_{kk})(\Delta S_{kk}(\mathbf{x},\boldsymbol{\sigma}) - S_{kl,kl}(\mathbf{x},\boldsymbol{\sigma})) - R_{mm}(\mathbf{x},\mathbf{Q}(\mathbf{F},\boldsymbol{\sigma})) = 0 \quad \text{in } \Omega, \tag{2.12}$$

$$(\delta_{ij} - Z_{ij})(\Delta S_{kk}(\mathbf{x},\boldsymbol{\sigma}) - S_{kl,kl}(\mathbf{x},\boldsymbol{\sigma})) - R_{mm}(\mathbf{x},\mathbf{Q}(\mathbf{F},\boldsymbol{\sigma}))_{,j} + R_{ik,k}(\mathbf{x},\mathbf{Q}(\mathbf{F},\boldsymbol{\sigma})) = 0 \quad \text{in } \Omega \tag{2.13}$$

Assuming $Z_{kk} \neq 2$, from (2.12) and (2.13) follows

$$R_{ik,k}(\mathbf{x},\mathbf{Q}(\mathbf{F},\boldsymbol{\sigma}) = 0 \quad \text{in } \Omega \tag{2.14}$$

Eq. (2.14) together with the boundary condition (2.10) give the boundary value problem for the vector field $\mathbf{q}$.

$$R_{ik,k}\left(\mathbf{x},\frac{1}{2}(\nabla\mathbf{q} + \nabla\mathbf{q}^{\mathrm{T}})\right) = 0 \quad \text{in } \Omega. \tag{2.15}$$

$$\boldsymbol{q} = 0 \quad \text{on } \partial\Omega, \tag{2.16}$$

If the boundary value problem (2.15) and (2.16) has a unique solution

$$\boldsymbol{q} \equiv \text{in } \Omega$$

then the equilibrium in the domain (2.7) holds and substituting (2.7) and (2.12) into (2.9) one obtains (2.6). For example, the system (2.15), (2.16) has a unique solution in the case

$$R_{kl}(\mathbf{x},\mathbf{Q})Q_{kl} \geqslant 0 \text{ or } R_{kl}(\mathbf{x},\mathbf{Q})Q_{kl} \leqslant 0 \text{ and } R_{kl}(\mathbf{x},\mathbf{Q})Q_{kl} = 0 \quad \text{if and only if } \mathbf{Q} = 0.$$

3. The problem of traction for spherically uniform anisotropic linear elastic solids in terms of stresses

The constitutive equation for spherically uniform anisotropic linear elastic solid is (Ting, 1999)

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}, \tag{3.1}$$

where

$$\boldsymbol{\sigma} = (\tau_{rr}, \tau_{\theta\theta}, \tau_{\varphi\varphi}, \tau_{\theta\varphi}, \tau_{\varphi r}, \tau_{r\theta})^{\mathrm{T}},$$
$$\boldsymbol{\varepsilon} = (\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{\varphi\varphi}, 2\varepsilon_{\theta\varphi}, 2\varepsilon_{\varphi r}, 2\varepsilon_{r\theta})^{\mathrm{T}},$$

and $\boldsymbol{C}$ is a positive definite constant symmetric 6×6 matrix.

Note that for such materials the constitutive laws (2.1) and (2.5) can be written in tensor form as

$$\sigma_{ij}(\mathbf{x}) = C_{ij}(\mathbf{x}, \boldsymbol{\varepsilon}) = C_{ijkl}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)\varepsilon_{kl}(\mathbf{x}), \tag{3.2}$$

$$\varepsilon_{ij}(\mathbf{x}) = S_{ij}(\mathbf{x}, \boldsymbol{\varepsilon}) = S_{ijkl}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)\sigma_{kl}(\mathbf{x}). \tag{3.3}$$

In Eq. (2.6) we can set

$$\begin{aligned} R_{ij}(\mathbf{x}, \mathbf{Q}) &= M(Q_{ij} + \alpha Q_{kk}\delta_{ij}) \\ Z_{ij} &= z\delta_{ij}, \end{aligned} \tag{3.4}$$

where the constant parameters M, z, α satisfy the inequalities

$$M \neq 0, \quad z \neq \frac{2}{3}, \quad -1 < \alpha < +\infty \tag{3.5}$$

in order for the systems (2.6), (2.7), (2.8) and (2.9), (2.10), (2.11) to be equivalent.

4. Spherically symmetric deformations for spherically uniform anisotropic linear elastic solids

The general form of the compliance matrix in (3.1) that allows spherically symmetric deformation is (Ting, 1999)

$$\boldsymbol{C} = \begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{12} & C_{23} & C_{22} & -C_{24} & -C_{25} & -C_{26} \\ 0 & C_{24} & -C_{24} & C_{44} & C_{45} & C_{46} \\ 0 & C_{25} & -C_{25} & C_{45} & C_{55} & C_{56} \\ 0 & C_{26} & -C_{26} & C_{46} & C_{56} & C_{66} \end{pmatrix}. \tag{4.1}$$

For spherically symmetric deformations the stress and strain tensors are defined by only two scalar functions depending on the radius

$$\sigma_{ij}(\mathbf{x}) = f(r)\delta_{ij} + g(r)\frac{x_i x_j}{r^2}, \tag{4.2}$$

$$\varepsilon_{ij}(\mathbf{x}) = F(r)\delta_{ij} + G(r)\frac{x_i x_j}{r^2}, \tag{4.3}$$

where

$$f = \tau_{\theta\theta} = \tau_{\varphi\varphi}, \quad g = \tau_{rr} - \tau_{\varphi\varphi},$$

$$F = \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi}, \quad G = \varepsilon_{rr} - \varepsilon_{\varphi\varphi}.$$

Hooke's law (3.1) and (4.1) gives

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} C_{12}+C_{22}+C_{23} & C_{12} \\ C_{11}+C_{12}-C_{22}-C_{23} & C_{11}-C_{12} \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}. \tag{4.4}$$

Due to the positive definiteness of the matrix $\boldsymbol{C}$ in (4.1) the constant coefficients entering (4.4) satisfy the system of inequalities

$$C_{22} > 0,$$
$$C_{22}^2 - C_{23}^2 > 0,$$
$$(C_{22}-C_{23})\lfloor C_{11}(C_{22}+C_{23}) - 2C_{12}^2 \rfloor > 0,$$

which can be reduced to

$$C_{22}+C_{23} > 0 \tag{4.5}$$

$$C_{22}-C_{23} > 0 \tag{4.6}$$

$$C_{11}(C_{22}+C_{23}) - 2C_{12}^2 > 0 \tag{4.7}$$

Substituting the tensors (4.2) and (4.3) into Eq. (2.9) we obtain the system of two second ordinary differential equations for f and g

$$R(D)\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.8}$$

where

$$R(D) = \left[\begin{pmatrix} D^2-2D & -D+2 \\ D^2+2D & -D-2 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ -2D^2-2D & 2D+2 \end{pmatrix}\right] \times \begin{pmatrix} C_{12}+C_{22}+C_{23} & C_{12} \\ C_{11}+C_{12}-C_{22}-C_{23} & C_{11}-C_{12} \end{pmatrix}^{-1}$$
$$+ M\begin{pmatrix} 2D^2-4D & 2D^2-8 \\ (2\alpha+1)D^2+(2\alpha+3)D & (2\alpha+1)D^2+(6\alpha+5)D+(4\alpha+6) \end{pmatrix},$$
$$D = r\frac{\mathrm{d}}{\mathrm{d}r}.$$

The characteristic polynomial of the system (4.8) is

$$\det R(D) = \frac{2(\alpha+1)M(3z-2)}{C_{11}(\eta-2\eta^2+\gamma)}(D-2)(D+1)(D^2+3D+2-2\gamma), \tag{4.9}$$

where γ and η are material parameters as introduced in Ting (1999)

$$\gamma = \frac{C_{22}+C_{23}-C_{12}}{C_{11}}, \quad \eta = \frac{C_{12}}{C_{11}}.$$

Since the parameters M, z, α are chosen to satisfy inequalities (3.5) the numerator in the right hand side of (4.9) does not vanish. The denominator is not equal to zero due to the inequalities (4.5), (4.6), (4.7).

The general solution of the system (4.8) is

$$\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = A_1 r^{-1} X_1 + A_2 r^2 X_2 + A_3 r^{(-1+3k)/2} X_3 + A_4 X_4 r^{(-1-3k)/2}, \tag{4.10}$$

where A_1, A_2, A_3, A_4 are arbitrary constants,

$$X_1 = \begin{pmatrix} 32C_{11}M\eta^2 + \eta(8 - 16C_{11}M) - (-1 + 9k^2)(-1 + 2C_{11}M) \\ 1 + 8\eta - 9k^2 + 2C_{11}M((1 - 4\eta)^2 - 9k^2) \end{pmatrix},$$
$$X_2 = \begin{pmatrix} -1 + 32\eta + 9k^2 - 2C_{11}M((1 - 4\eta^2) - 9k^2)(-1 + 3k) \\ 25 - 16\eta - 9k^2 + C_{11}M((1 - 4\eta)^2 - 9k^2)(-1 + 3k) \end{pmatrix},$$
$$X_3 = \begin{pmatrix} -1 + 3k \\ -3 + 3k \end{pmatrix},$$
$$X_4 = \begin{pmatrix} 1 - 3k \\ 3 + 3k \end{pmatrix},$$
$$k = \frac{1}{3}(1 + 8\gamma)^{1/2}.$$

We may note here that the general solution depends on the parameter M. Eq. (2.9) on their own do not give solutions of elasticity problems and one has to take into account the boundary conditions (2.10).

For example, consider a spherical shell $R_1 < r < R_2$ under a uniform internal and external pressure. Then, together with the boundary conditions of traction

$$\sigma_{ij}(\mathbf{x})n_j(\mathbf{x})|_{r=R_1} = -p_1 n_i(\mathbf{x})$$
$$\sigma_{ij}(\mathbf{x})n_j(\mathbf{x})|_{r=R_2} = -p_2 n_i(\mathbf{x})$$

that take the form

$$(f + g)|_{r=R_1} = -p_1, \tag{4.11}$$

$$(f + g)|_{r=R_2} = -p_2, \tag{4.12}$$

one imposes equilibrium at the boundary (2.10) that in the case of a spherically symmetric deformation can be written as

$$\left(f' + g' + \frac{2}{r}g\right)\Big|_{r=R_1} = 0, \tag{4.13}$$

$$\left(f' + g' + \frac{2}{r}g\right)\Big|_{r=R_2} = 0, \tag{4.14}$$

Substituting (4.10) into the boundary conditions (4.11), (4.12), (4.13), (4.14) we obtain the values of the constants A_1, A_2, A_3, A_4.

The solution is

$$A_1 = A_2 = 0,$$
$$A_3 = \frac{p_2 R_2^{3(1+k)/2} - p_1 R_1^{3(1+k)/2}}{4(R_2^{3k} - R_1^{3k})},$$
$$A_4 = \frac{p_2 R_1^{3k} R_2^{3(1+k)/2} - p_1 R_1^{3(1+k)/2} R_2^{3k}}{4(R_2^{3k} - R_1^{3k})}.$$

Then the stresses are given by the formulae

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = f = -p_2 \frac{q^+ r^{3k} - q^- R_1^{3k}}{R_2^{3k} - R_1^{3k}} \left(\frac{R_2}{r}\right)^{3(1+k)/2} - p_1 \frac{q^- R_2^{3k} - q^+ r^{3k}}{R_2^{3k} - R_1^{3k}} \left(\frac{R_1}{r}\right)^{3(1+k)/2},$$
$$\sigma_{rr} = f + g = -p_2 \frac{r^{3k} - R_1^{3k}}{R_2^{3k} - R_1^{3k}} \left(\frac{R_2}{r}\right)^{3(1+k)/2} - p_1 \frac{R_2^{3k} - r^{3k}}{R_2^{3k} - R_1^{3k}} \left(\frac{R_1}{r}\right)^{3(1+k)/2},$$

where $q^{\pm} = \frac{1}{4}(1 \pm 3k)$, which are in agreement with whose in Lekhnitskii (1963) and Ting (1999).

5. Conclusion

We have applied the formulation of the boundary value problem in terms of stresses in 3D to inhomogeneous anisotropic elastic solids, specifically spherically uniform anisotropic solids and solved an example for a spherical shell. By this method the stresses are obtained directly without the necessity of differentiation of the displacements as in the standard displacement formulation of 3D boundary value problems of elasticity.

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